Math 4550

Topic 3 - Direct Products

Def: Let G_1, G_2 be groups.
The direct product of G_1 and G_2 is

$$G_1 \times G_2 = \{ (x_1, x_2) \mid x_1 \in G_1 \text{ and } x_2 \in G_2 \}$$

Ex: $U_3 = \{1, \rho, \rho^2\}$

$\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$

$U_3 \times \mathbb{Z}_2 = \{(1,\bar{0}), (1,\bar{1}), (\rho,\bar{0}), (\rho,\bar{1}), (\rho^2,\bar{0}), (\rho^2,\bar{1})\}$

<u>Theorem:</u> Let G_1 and G_2 be groups with identity elements e_1 and e_2, respectively. The direct product $G_1 \times G_2$ is a group under the operation

$$(a,b)(c,d) = (\underbrace{ac}_{\text{operation in } G_1}, \underbrace{bd}_{\text{operation in } G_2})$$

The identity element is (e_1, e_2).
The inverse of (a,b) is (a^{-1}, b^{-1}).

<u>proof:</u>

① (closure) Let $(a,b), (c,d) \in G_1 \times G_2$.
Then $a, c \in G_1$ and $b, d \in G_2$.
Since G_1 is a group we get $ac \in G_1$.
Since G_2 is a group we get $bd \in G_2$.
Thus, $(a,b)(c,d) = (ac, bd) \in G_1 \times G_2$.

② Let $(a,b), (c,d), (f,g) \in G_1 \times G_2$.

Then,
$$(a,b)\big[(c,d)(f,g)\big] = (a,b)\big[(cf, dg)\big]$$

$$= (a(cf), b(dg))$$

$$= ((ac)f, (bd)g)$$

Since G_1 and G_2 are groups and are associative

$$= [(ac, bd)](f, g)$$

$$= [(a,b)(c,d)](f,g)$$

Thus, $G_1 \times G_2$ is associative.

③ (identity)

Let $(a,b) \in G_1 \times G_2$.

Then,

$$(e_1, e_2)(a,b) = (e_1 a, e_2 b) = (a,b)$$

$$(a,b)(e_1,e_2) = (ae_1, be_2) = (a,b)$$

since e_1 is the identity of G_1 and e_2 is the identity of G_2

Thus, (e_1, e_2) is an identity for $G_1 \times G_2$.

④ Let $(a,b) \in G_1 \times G_2$.

Then $a \in G_1$ and $b \in G_2$.

Since G_1 is a group we get that $a^{-1} \in G_1$

Since G_2 is a group we get that $b^{-1} \in G_2$.

Thus, $(a^{-1}, b^{-1}) \in G_1 \times G_2$ and we have:

$$(a,b)(a^{-1},b^{-1}) = (aa^{-1}, bb^{-1}) = (e_1, e_2)$$
$$(a^{-1},b^{-1})(a,b) = (a^{-1}a, b^{-1}b) = (e_1, e_2)$$

So, (a^{-1}, b^{-1}) is the inverse of (a,b).

By 1, 2, 3, 4 we get that $G_1 \times G_2$ is a group. $\blacksquare$

Ex:

$U_3 = \{1, \rho, \rho^2\}$ where $\rho^3 = 1$ ← group under multiplication

$\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ ← group under addition

$$U_3 \times \mathbb{Z}_2 = \{\underbrace{(1,\bar{0})}_{\text{identity element}}, (1,\bar{1}), (\rho,\bar{0}), (\rho,\bar{1}), (\rho^2,\bar{0}), (\rho^2,\bar{1})\}$$

Sample calculations:

$$(\rho,\bar{1})(\rho^2,\bar{0}) = (\rho\cdot\rho^2, \bar{1}+\bar{0}) = (\rho^3,\bar{1}) = (1,\bar{1})$$

operation in U_3

operation in $\mathbb{Z}_2$

$$(1,\bar{0})(\rho^2,\bar{1}) = (1\cdot\rho^2, \bar{0}+\bar{1}) = (\rho^2,\bar{1})$$

The theorem says that

$(\rho^2)^{-1} = \rho$ in U_3 since $\rho^2\cdot\rho = 1$

$$(\rho^2,\bar{1})^{-1} = ((\rho^2)^{-1}, \bar{1}^{-1}) = (\rho, \bar{1})$$

$\bar{1}^{-1} = \bar{1}$ in $\mathbb{Z}_2$ since $\bar{1}+\bar{1}=\bar{0}$

Let's verify it:

$$(\rho^2,\bar{1})\cdot(\rho,\bar{1}) = (\rho^2\cdot\rho, \bar{1}+\bar{1}) = (\rho^3,\bar{0})$$

$$= \underbrace{(1,\bar{0})}_{\text{identity in } U_3\times\mathbb{Z}_2}$$

Ex: $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\underbrace{(\bar{0},\bar{0})}_{\substack{\text{identity}\\ \text{element}}}, (\bar{0},\bar{1}), (\bar{1},\bar{0}), (\bar{1},\bar{1})\}$

Since both groups use addition instead of writing $(\bar{0},\bar{1})(\bar{1},\bar{0}) = (\bar{0}+\bar{1}, \bar{1}+\bar{0}) = (\bar{1},\bar{1})$ we write $(\bar{0},\bar{1})+(\bar{1},\bar{0}) = (\bar{0}+\bar{1}, \bar{1}+\bar{0}) = (\bar{1},\bar{1})$.

Here's the group table:

$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$
$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$
$(\bar{1},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{1},\bar{1})$	$(\bar{0},\bar{1})$
$(\bar{0},\bar{1})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$
$(\bar{1},\bar{1})$	$(\bar{1},\bar{1})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{0})$

Some sample calculations are:

$$(\bar{1},\bar{1}) + (\bar{1},\bar{0}) = (\bar{1}+\bar{1}, \bar{1}+\bar{0}) = (\bar{0},\bar{1})$$

$$(\bar{1},\bar{0}) + (\bar{1},\bar{0}) = (\bar{1}+\bar{1}, \bar{0}+\bar{0}) = (\bar{0},\bar{0})$$

You can see from the table that the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian.

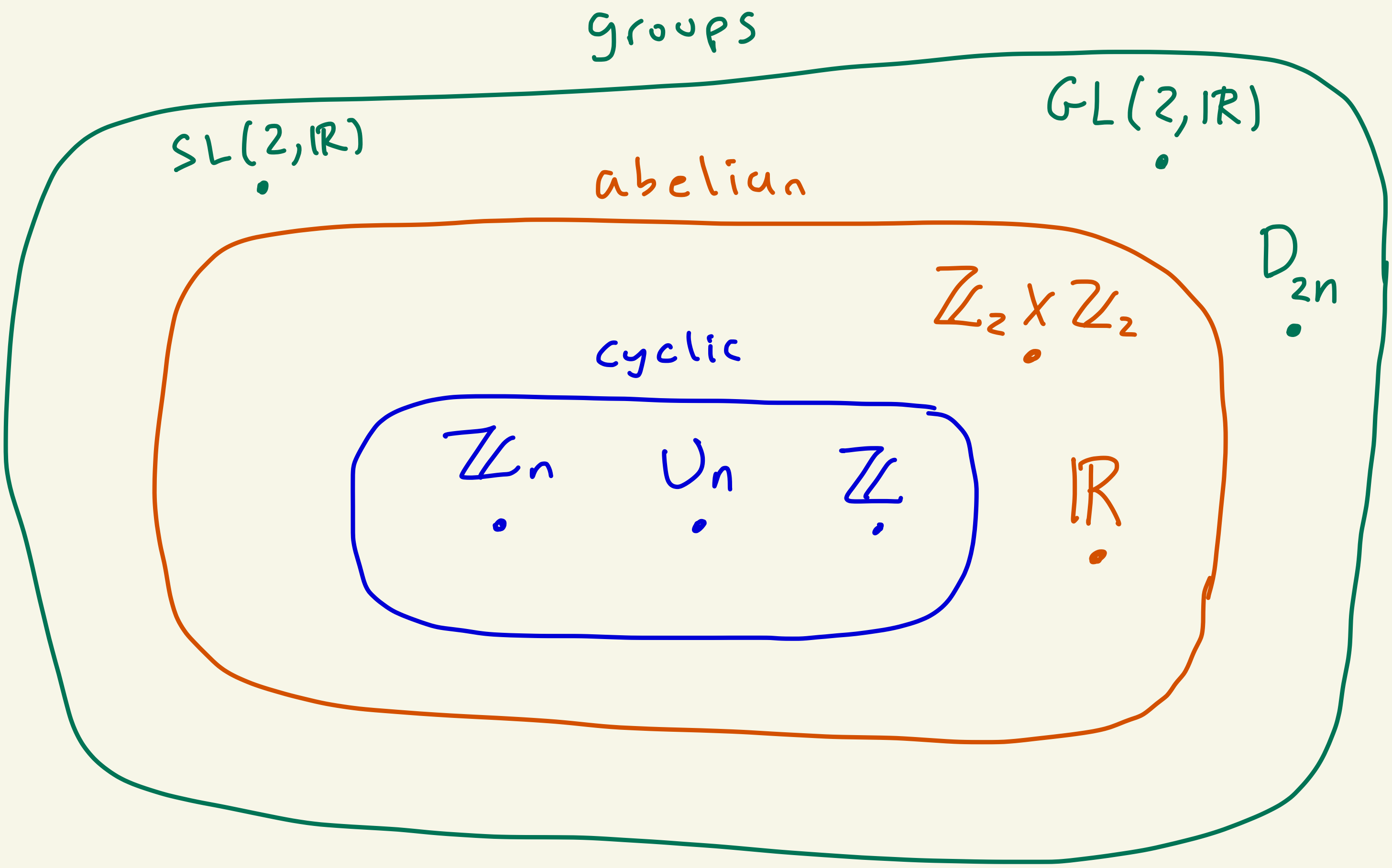
However it is not cyclic.

$$\left.\begin{aligned}
\langle(\bar{0},\bar{0})\rangle &= \{(\bar{0},\bar{0})\} \\
\langle(\bar{1},\bar{0})\rangle &= \{(\bar{1},\bar{0}),(\bar{0},\bar{0})\} \\
\langle(\bar{0},\bar{1})\rangle &= \{(\bar{0},\bar{1}),(\bar{0},\bar{0})\} \\
\langle(\bar{1},\bar{1})\rangle &= \{(\bar{1},\bar{1}),(\bar{0},\bar{0})\}
\end{aligned}\right\}$$

no element generates all of $\mathbb{Z}_2 \times \mathbb{Z}_2$ so $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic.

Thus, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian but not cyclic.

Picture of the world of groups that we have so far:



<u>Theorem:</u> If G_1 and G_2 are both abelian groups, then $G_1 \times G_2$ is abelian.

<u>proof:</u> HW $\boxtimes$

<u>Ex:</u> $\mathbb{Z}_n \times \mathbb{Z}_m$ is abelian

<u>Theorem:</u> $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $\gcd(m,n)=1$.

<u>proof:</u> (Don't do in class, point out in notes)

($\Leftarrow$) Suppose $\gcd(m,n)=1$.

We will show that $\mathbb{Z}_m \times \mathbb{Z}_n = \langle (\bar{1},\bar{1}) \rangle$.

Suppose that

$$\underbrace{(\bar{1},\bar{1})+(\bar{1},\bar{1})+\dots+(\bar{1},\bar{1})}_{d \text{ times}} = (\bar{0},\bar{0})$$

where $d>0$.

Then, $(\bar{d},\bar{d})=(\bar{0},\bar{0})$.

So, $\bar{d}=\bar{0}$ in $\mathbb{Z}_m$ and $\bar{d}=\bar{0}$ in $\mathbb{Z}_n$.

Thus, m divides d and n divides d.

So, d is a common multiple of m and n.

From number theory, the least common multiple of m and n is $\frac{mn}{\gcd(m,n)}$ which in this case is mn.

Thus, $mn \le d$.

So the order of $(\bar{1},\bar{1})$ is at least mn.

Also,

$$\underbrace{(\bar{1},\bar{1})+(\bar{1},\bar{1})+\dots+(\bar{1},\bar{1})}_{mn \text{ times}} = (\underbrace{mn}_{\bar{0} \text{ in } \mathbb{Z}_m}, \underbrace{mn}_{\bar{0} \text{ in } \mathbb{Z}_n}) = (\bar{0},\bar{0})$$

Thus, $(\bar{1},\bar{1})$ has order mn.

So, $\mathbb{Z}_m \times \mathbb{Z}_n = \langle (\bar{1},\bar{1}) \rangle$ since $|\mathbb{Z}_m \times \mathbb{Z}_n| = mn$.

$(\Rightarrow)$ Suppose $d = \gcd(m,n) > 1$.

Let $(\bar{r},\bar{s}) \in \mathbb{Z}_m \times \mathbb{Z}_n$.

Then,

$$\underbrace{(\bar{r},\bar{s})+(\bar{r},\bar{s})+\dots+(\bar{r},\bar{s})}_{\frac{mn}{d} \text{ times}} = \left(\overline{\tfrac{mn}{d} r}, \overline{\tfrac{mn}{d} s}\right)$$

(note: $d|m$ & $d|n$ so $\frac{mn}{d} \in \mathbb{Z}$)

$\bar{m} = \bar{0}$ in $\mathbb{Z}_m$

$\bar{n} = \bar{0}$ in $\mathbb{Z}_n$

$$= \left(\tfrac{n}{d}\bar{m}\bar{r}, \tfrac{m}{d}\bar{n}\bar{s}\right) = (\bar{0},\bar{0})$$

($\frac{n}{d}, \frac{m}{d} \in \mathbb{Z}$ since $d|m$, $d|n$)

So, every element of $\mathbb{Z}_m \times \mathbb{Z}_n$ has order at most $\frac{mn}{d} < mn$ since $d > 1$. So, $\mathbb{Z}_m \times \mathbb{Z}_n$ is not cyclic if $\gcd(m,n) > 1$. $\square$